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On positive definite solutions of the family of matrix equations $X + A^*X^{-n}A = Q$

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Abstract

The nonlinear matrix equation $X + A^*X^{-n}A = Q$ and properties of its positive definite solutions are studied. Sufficient conditions for existence the minimal X_S and special X_1 positive definite solutions are derived and iterative procedures for computing these solution are discussed.

MSC: 15A24; 15A45; 65F35

Keywords: Nonlinear matrix equation; Positive definite solution

1. Introduction

Consider the nonlinear matrix equation

$$X + A^*X^{-n}A = Q, \tag{1}$$

where Q is an $m \times m$ positive definite matrix, A is an arbitrary nonsingular matrix, n is a positive integer, bigger than 1. We introduce the corresponding matrix function

$$G(X) = \sqrt[n]{A(Q-X)^{-1}A^*}. \tag{2}$$

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This equation and properties of its positive definite solutions have been investigated in some special cases: in [4,6] the case $Q = I$ and $n \geqslant 1$ is investigated. The more general case $X^s \pm A^{\mathrm{T}} X^{-t} A = I$ is analyzed by Liu and Gao [9] and Du and Hou [2]. In [5] the equation $X \pm A^* X^{-q} A = Q$, where $q \in (0, 1]$ is considered.

Some properties of positive definite solutions of (1) are derived in [7,8]. In this paper we continue to investigate the properties of positive definite solutions of (1). Let X_{S} and X_{L} be positive definite solutions of Eq. (1). If every positive definite solution X satisfies $X_{\mathrm{S}} \leqslant X \leqslant X_{\mathrm{L}}$, then X_{S} and X_{L} are minimal and maximal solutions of (1), respectively. It is well known [7,4] that under additional restrictions Eq. (1) has a special positive definite solution X_1 with the property $\|X_1^{-1}\| < \frac{n+1}{n} \|Q^{-1}\|$. If there is a largest positive definite solution X_{L} of (1), then $X_{\mathrm{L}} \equiv X_1$ [7,4].

We study the properties of the matrix sequence $X_{k+1} = G(X_k)$ for suitable chosen starting value X_0. In this paper we describe starting values for computing a positive definite solution of (1), when A is nonsingular. The rates of convergence for the proposed starting matrices X_0 depend on one parameter α or β that is derived from the singular values of $Q^{-\frac{n}{2}}\, A\, Q^{-\frac{1}{2}}$.

We show if $\|A Q^{-\frac{1}{2}}\|^2 \|Q^{-1}\|^n < n^n/(n+1)^{n+1}$ for the spectral norm $\|.\|$, then there exists a fixed point of $G(X)$ on the set of Hermitian positive definite matrices X with property $\|X\| \leqslant n/(n+1)1/\|Q^{-1}\|$. If this fixed point is unique then it is the smallest positive definite solution X_{S} of (1).

We use $\|A\|_F$ to denote the Frobenius norm of a matrix A. The symbol $\|A\|$ stands for the spectral norm of a matrix A, i.e., $\|A\| = \sigma_1(A)$, where $\sigma_1(A) \geqslant \sigma_2(A) \geqslant \cdots \geqslant \sigma_m(A) \geqslant 0$ are the singular values of A in non increasing order. If A is a Hermitian positive semidefinite matrix we use the eigenvalues $\lambda_i(A)$ of A where $\lambda_1(A) \geqslant \lambda_2(A) \geqslant \cdots \geqslant \lambda_m(A) \geqslant 0$. For the second vector norm we use the notation $\|x\|_2 = \sqrt{x^* x}$. Throughout the paper we use the fact that $\lambda_m(A) I \leqslant A$ and $A \leqslant \lambda_1(A) I$ in our partial order. The set of all $m \times m$ Hermitian positive definite matrices will be denoted by $\mathscr{P}(m)$.

In our investigation we use the properties of the function $\varphi(x) = x^n(1-x)$ where $x \in [0, 1]$. This function has maximum at the point $n/(n+1)$ and it is monotonically increasing on $[0, n/(n+1)]$ and monotonically decreasing on $[n/(n+1), 1]$.

We utilize the vector representations of stacking the columns of a matrix in one vector, obtaining $x = \mathrm{vec}(X)$, where X is a $p \times q$ matrix and x is a pq vector column. In particular, we have $\mathrm{vec}(AXB) = (B^{\mathrm{T}} \otimes A)\, \mathrm{vec}(X)$ and $\|\mathrm{vec}(X)\|_2 = \|X\|_F$. As a corollary we obtain

$$\|AX\|_F = \|\mathrm{vec}(AX)\|_2 = \|(I \otimes A)\mathrm{vec}(X)\|_2 \leqslant \|I \otimes A\|_2 \|\mathrm{vec}(X)\|_2 = \|A\|_2 \|X\|_F.$$

In presentation of the results we use two kind estimates for the difference $\|G(X) - G(Y)\|$. The first kind is based of Bhatia's theorem (Theorem X.3.8, [1]) and second kind is next Theorem 2 which is derived for the matrix function $G(X)$.

Theorem 1 (*Theorem X.3.8, Bhatia [1]*). *Let f be a monotone function on $(0, \infty)$ and let A, B be two positive operators bounded below by a, i.e. $A > aI$ and $B > aI$ for a positive number a. If there exists $f'(a)$, then for every unitary invariant norm*

$$|||f(A) - f(B)||| < f'(a)|||A - B|||.$$

Theorem 2. *Consider the matrix function $G(X)$ where X is a positive definite matrix and there exists a real number $\beta < 1$ such that $X \leqslant \beta Q$. Then*

$$\|G(X) - G(Y)\|_F \leqslant \frac{1}{n}\left(\frac{\lambda_1(AQ^{-1}A^*)}{1-\beta}\right)^{(n+1)/n} \|A^{-1}\|^2 \|X - Y\|_F$$

for all $X, Y \in (0, \beta Q]$.

Proof. We take $R^{-1} = A(Q-X)^{-1}A^*$ and $S^{-1} = A(Q-Y)^{-1}A^*$ and then

$$G(X) - G(Y) = \sqrt[n]{R^{-1}} - \sqrt[n]{S^{-1}}.$$

Since $X = Q - A^*RA$ and $Y = Q - A^*SA$ we have

$$X - Y = A^*(S-R)A.$$

Using the vec representation we obtain

$$\begin{aligned} \mathrm{vec}(G(X) - G(Y)) &= \mathrm{vec}\left(\sqrt[n]{R^{-1}} - \sqrt[n]{S^{-1}}\right), \\ \mathrm{vec}(X - Y) &= \mathrm{vec}(A^*(S-R)A). \end{aligned} \tag{3}$$

For positive definite matrices R and S we know the identity

$$S - R = \sum_{i=1}^{n} \sqrt[n]{R^i}\left(\sqrt[n]{R^{-1}} - \sqrt[n]{S^{-1}}\right)\sqrt[n]{S^{n+1-i}}.$$

We obtain

$$X - Y = A^*(S-R)A = \sum_{i=1}^{n} A^*\sqrt[n]{R^i}\left(\sqrt[n]{R^{-1}} - \sqrt[n]{S^{-1}}\right)\sqrt[n]{S^{n+1-i}}A.$$

The vec representation of the last equality leads to

$$\begin{aligned} \mathrm{vec}(X - Y) &= \left(\sum_{i=1}^{n}\left(\sqrt[n]{S^{n+1-i}}A\right)^{\mathrm{T}} \otimes A^*\sqrt[n]{R^i}\right)\mathrm{vec}\left(\sqrt[n]{R^{-1}} - \sqrt[n]{S^{-1}}\right) \\ &= (A^{\mathrm{T}} \otimes A^*)\left(\sum_{i=1}^{n}\left(\sqrt[n]{S^{n+1-i}}\right)^{\mathrm{T}} \otimes \sqrt[n]{R^i}\right)\mathrm{vec}\left(\sqrt[n]{R^{-1}} - \sqrt[n]{S^{-1}}\right). \end{aligned}$$

According to (3) we receive

$$\mathrm{vec}(X - Y) = (A^{\mathrm{T}} \otimes A^*)\left(\sum_{i=1}^{n}\left(\sqrt[n]{S^{n+1-i}}\right)^{\mathrm{T}} \otimes \sqrt[n]{R^i}\right)\mathrm{vec}(G(X) - G(Y))$$

$$\mathrm{vec}(G(X) - G(Y)) = \left(\sum_{i=1}^{n}\left(\sqrt[n]{S^{n+1-i}}\right)^{\mathrm{T}} \otimes \sqrt[n]{R^i}\right)^{-1}(A^{\mathrm{T}} \otimes A^*)^{-1}\mathrm{vec}(X - Y).$$

We evaluate the vector norm induces the spectral matrix norm:

$$\|\mathrm{vec}(G(X)-G(Y))\|_2 \leqslant \left\| \left(\sum_{i=1}^{n} \left(\sqrt[n]{S^{n+1-i}} \right)^{\mathrm{T}} \otimes \sqrt[n]{R^i} \right)^{-1} \right\| \times \|(A^{\mathrm{T}} \otimes A^*)^{-1}\| \|\mathrm{vec}\,(X-Y)\|_2.$$

Since $X < \beta Q$ and $Y < \beta Q$ then

$$R = A^{-*}(Q-X)A^{-1} > (1-\beta)\lambda_m(A^{-*}QA^{-1})I = \frac{1-\beta}{\lambda_1(AQ^{-1}A^*)}\, I,$$
$$S = A^{-*}(Q-Y)A^{-1} > \frac{1-\beta}{\lambda_1(AQ^{-1}A^*)}\, I.$$

We denote $\mu = 1-\beta/\lambda_1(AQ^{-1}A^*)$ and then we write

$$\sqrt[n]{R^i} > \sqrt[n]{\mu^i}\, I, \quad \sqrt[n]{S^{n+1-i}} > \sqrt[n]{\mu^{n+1-i}}\, I,$$
$$\lambda_m\left(\sqrt[n]{R^i}\right) > \sqrt[n]{\mu^i}, \quad \lambda_m\left(\sqrt[n]{S^{n+1-i}}\right) > \sqrt[n]{\mu^{n+1-i}}.$$

Thus

$$\lambda\left(\sum_{i=1}^{n} \left(\sqrt[n]{S^{n+1-i}} \right)^{\mathrm{T}} \otimes \sqrt[n]{R^i} \right) = \sum_{i=1}^{n} \lambda\left(\sqrt[n]{S^{n+1-i}} \right) \lambda\left(\sqrt[n]{R^i} \right),$$
$$\lambda_m\left(\sum_{i=1}^{n} \left(\sqrt[n]{S^{n+1-i}} \right)^{\mathrm{T}} \otimes \sqrt[n]{R^i} \right) > \sum_{i=1}^{n} \sqrt[n]{\mu^{n+1-i}}\sqrt[n]{\mu^i} = n\mu\sqrt[n]{\mu}.$$

Hence

$$\left\| \left(\sum_{i=1}^{n} \left(\sqrt[n]{S^{n+1-i}} \right)^{\mathrm{T}} \otimes \sqrt[n]{R^i} \right)^{-1} \right\| \leqslant \frac{1}{n\mu\sqrt[n]{\mu}}.$$

Since $\|\mathrm{vec}(X-Y)\|_2 = \|X-Y\|_F$ we compute

$$\|G(X)-G(Y)\|_F \leqslant \frac{1}{n\mu\sqrt[n]{\mu}} \|(A^{\mathrm{T}} \otimes A^*)^{-1}\| \|X-Y\|_F.$$

But $\|A^{\mathrm{T}} \otimes A^*\| = \|A\|^2$ and A is nonsingular. Then

$$\|(A^{\mathrm{T}} \otimes A^*)^{-1}\| = \|(A^{-\mathrm{T}} \otimes A^{-*})\| = \|A^{-1}\|^2.$$

Hence

$$\|G(X)-G(Y)\|_F \leqslant \frac{1}{n(\mu)^{(n+1)/n}}\,\|A^{-1}\|^2\|X-Y\|_F$$
$$= \frac{1}{n}\left(\frac{\lambda_1(AQ^{-1}A^*)}{1-\beta}\right)^{(n+1)/n}\|A^{-1}\|^2\|X-Y\|_F.$$

The theorem is proved. □

We make a study of properties of a special solution X_1. For this we need the following theorem.

Theorem 3 (*Theorem 2.1, Furuta [3]*). *Let A and B be positive operators on a Hilbert space H such that* $M_1 I \geqslant A \geqslant m_1 I > 0$, $M_2 I \geqslant B \geqslant m_2 I > 0$ *and* $0 < A \leqslant B$. *Then*

$$A^t \leqslant \left(\frac{M_1}{m_1}\right)^{t-1} B^t$$

and

$$A^t \leqslant \left(\frac{M_2}{m_2}\right)^{t-1} B^t$$

hold for any $t \geqslant 1$.

In case when A and B are positive definite matrices we utilize the above inequalities where $m_2 I \leqslant B \leqslant M_2 I$. For example, we can take $M_2 = \|B\|$, $m_2 = \|B^{-1}\|^{-1}$.

2. Properties of the positive definite solutions

2.1. Existence a positive definite solution

We prove sufficient condition in order to exist a positive definite solution of (1). We investigate an iterative method and its properties. This method with different starting point defines two monotone matrix sequences. These sequences converge to a positive definite solution of (1).

We use the following theorems:

Theorem 4 (*Hasanov and Ivanov [7]*). *If Eq.* (1) *has a positive definite solution X, then*

$$\sqrt[n]{AQ^{-1}A^*} < X \leqslant Q - \frac{1}{(\|Q\|\|Q^{-1}\|)^{n-1}}\,A^*Q^{-n}A.$$

Theorem 5 (*Hasanov and Ivanov [7]*). *If the matrix equation* (1) *with nonsingular matrix A has a positive equation definite solution, then it has a minimal solution* X_S. *Moreover, the iterative algorithm* $X_k = G(X_{k-1})$ *with* $X_0 = \sqrt[n]{AQ^{-1}A^*}$ *converges to* X_S.

Let Q be a positive definite matrix. We define the set of Hermitian positive definite matrices

$$K = \left\{ X \in \mathscr{P}(m) : \|X\| \leqslant \frac{n}{n+1} \frac{1}{\|Q^{-1}\|} \right\}$$

and investigate the map $G(X)$ defined by (2).

Theorem 6. *If*

$$\|AQ^{-\frac{1}{2}}\|^2 \, \|Q^{-1}\|^n < \frac{n^n}{(n+1)^{n+1}}, \tag{4}$$

then there exists a fixed point for $G(X)$ on K. Moreover, if

$$\min(p, q) < 1,$$

where

$$p = \frac{1}{n} [(n+1)\lambda_1(AQ^{-1}A^*)]^{(n+1)/n} \|A^{-1}\|^2 \tag{5}$$

and

$$q = \frac{1}{n} \left(\frac{n}{n+1} \frac{\|A^{-1}\|}{\|Q^{-1}\|} \right)^2 [(n+1)\lambda_1(AQ^{-1}A^*)]^{(n-1)/n}, \tag{6}$$

then this fixed point is unique. The unique fixed point is a unique positive definite solution $\hat{X}$ on K of Eq. (1).

Proof. We will prove that the matrix function $G(X)$ defines a continuous map on the set of Hermitian positive definite matrices K.

It is obvious that K is a convex, closed and bounded set. Since $\|Q^{-1}X\| \leqslant \|Q^{-1}\| \, \|X\| \leqslant n/(n+1)$ for every $X \in K$ we obtain that $G(X) = \sqrt[n]{AQ^{-1}(I - Q^{-1}X)^{-1}A^*}$ is a continuous function on K. Besides

$$\begin{aligned} \|G(X)\| &\leqslant \sqrt[n]{\frac{\|AQ^{-1}A^*\|}{1 - \frac{n}{n+1}}} \\ &\leqslant (n+1)^{\frac{1}{n}} \|AQ^{-\frac{1}{2}}\|^{2/n} \\ &< \frac{n}{n+1} \frac{1}{\|Q^{-1}\|} \end{aligned}$$

we conclude that $G(X) \in K$. Thus $G(X)$ has a fixed point on K. From $X \in K$ we have $\|X\| \leqslant n/(n+1)1/\|Q^{-1}\|$ and then $X \leqslant n/(n+1)Q$. Assume there exist two different fixed points $X, Y \in K$. We obtain

$$\begin{aligned} \|G(X) - G(Y)\| &\leqslant \|G(X)(G(Y)^{-1} - G(Y)^{-1})G(Y)\| \\ &\leqslant \|G(X)\| \|G(Y)\| \|G(X)^{-1} - G(Y)^{-1}\| \\ &< \left(\frac{n}{n+1} \frac{1}{\|Q^{-1}\|} \right)^2 \|G(X)^{-1} - G(Y)^{-1}\|. \end{aligned}$$

We have

$$G(X)^{-1} = \sqrt[n]{A^{-*}(Q-X)A^{-1}} \geqslant \sqrt[n]{\frac{1}{n+1}\, A^{-*}QA^{-1}} \geqslant \sqrt[n]{\frac{\lambda_m(A^{-*}QA^{-1})}{n+1}}\, I,$$

$$G(Y)^{-1} = \sqrt[n]{A^{-*}(Q-Y)A^{-1}} \geqslant \sqrt[n]{\frac{\lambda_m(A^{-*}QA^{-1})}{n+1}}\, I.$$

Using Theorem 1 we obtain

$$\begin{aligned}
\|G(X)-G(Y)\| &\leqslant \left(\frac{n}{n+1}\,\frac{1}{\|Q^{-1}\|}\right)^2 \frac{1}{n}\left(\frac{\lambda_m(A^{-*}QA^{-1})}{n+1}\right)^{(1-n)/n} \|A^{-1}\|^2\|X-Y\| \\
&= \frac{1}{n}\left(\frac{n}{n+1}\,\frac{\|A^{-1}\|}{\|Q^{-1}\|}\right)^2 \left(\frac{n+1}{\lambda_m(A^{-*}QA^{-1})}\right)^{(n-1)/n} \|X-Y\| \\
&= \frac{1}{n}\left(\frac{n}{n+1}\,\frac{\|A^{-1}\|}{\|Q^{-1}\|}\right)^2 ((n+1)\lambda_1(AQ^{-1}A^*))^{(n-1)/n}\|X-Y\| \\
&= q\|X-Y\|.
\end{aligned}$$

If $q<1$, then

$$\|G(X)-G(Y)\| \leqslant q\|X-Y\| < \|X-Y\|.$$

Using Theorem 2, if $p<1$, we obtain

$$\begin{aligned}
\|G(X)-G(Y)\|_F &\leqslant \frac{1}{n}\left(\frac{\lambda_1(AQ^{-1}A^*)}{1-\beta}\right)^{(n+1)/n} \|A^{-1}\|^2\|X-Y\|_F \\
&\leqslant p\|X-Y\|_F < \|X-Y\|_F.
\end{aligned}$$

Hence, if $\min(p,q)<1$, then the map $G(X)$ is a contraction on K and its fixed point is unique. The fixed point can be found using the matrix sequence

$$X_{s+1} = G(X_s), \quad s=0,1,\ldots \tag{7}$$

for every $X_0 \in K$. From $\hat{X} = G(\hat{X})$ it follows that $\hat{X}$ is a positive definite solution of (1).

The factor p characterizes the rate of convergence of (7) if $p<1$. The factor q characterizes the rate of convergence of (7) if $q<1$. □

Corollary 7. *If the inequality* (4) *holds, then* $\min(p,q)=p$, *where* p *and* q *are defined by* (5) *and* (6), *respectively.*

Proof. Using the inequality (4) we have

$$\lambda_1(AQ^{-1}A^*) \leqslant \|AQ^{-1}A^*\| = \|AQ^{-1/2}\|^2 < \frac{n^n}{(n+1)^{n+1}} \frac{1}{\|Q^{-1}\|^n},$$

$$\lambda_1(AQ^{-1}A^*) < \frac{n^n}{(n+1)^{n+1}} \frac{1}{\|Q^{-1}\|^n},$$

$$\left(\frac{n}{n+1} \frac{1}{\|Q^{-1}\|}\right)^2 \left(\frac{1}{(n+1)\,\lambda_1(AQ^{-1}A^*)}\right)^{2/n} > 1,$$

$$\left(\frac{n}{n+1} \frac{1}{\|Q^{-1}\|}\right)^2 [(n+1)\lambda_1(AQ^{-1}A^*)]^{(n-1-n-1)/n} > 1,$$

$$\frac{\frac{1}{n}\left(\frac{n}{n+1} \frac{\|A^{-1}\|}{\|Q^{-1}\|}\right)^2 \left(\frac{n+1}{\lambda_m(A^{-*}QA^{-1})}\right)^{(n-1)/n}}{\frac{1}{n}[(n+1)\lambda_1(AQ^{-1}A^*)]^{(n+1)/n}\|A^{-1}\|^2} > 1,$$

$$q > p. \qquad \square$$

Corollary 8. *If there exists a unique fixed point $\hat{X}$ of $G(X)$ on K then this fixed point is the smallest positive definite solution of* (1), *i.e.*, $\hat{X} \equiv X_S$.

Proof. We shall prove that $\hat{X} \equiv X_S$. According to Theorem 5 we know that the matrix sequence $X_k = G(X_{k-1})$ with $X_0 = \sqrt[n]{AQ^{-1}A^*}$ converges to X_S. Since

$$\|X_0\| = \left\|\sqrt[n]{AQ^{-1}A^*}\right\| \leqslant \|AQ^{-1/2}\|^{2/n} < \left(\frac{n^n}{(n+1)^{n+1}} \frac{1}{\|Q^{-1}\|^n}\right)^{1/n} < \frac{n}{n+1} \frac{1}{\|Q^{-1}\|}.$$

The last inequality means that $X_0 \in K$, i.e., $\hat{X} \equiv X_S$. $\square$

Further on, we shall consider the new restriction for the matrices A, Q. This new condition is obtained from inequality (4). Using condition (4) we have

$$\|Q^{-\frac{n}{2}}AQ^{-1}A^*Q^{-\frac{n}{2}}\| \leqslant \|AQ^{-\frac{1}{2}}\|^2\|Q^{-1}\|^n < \frac{n^n}{(n+1)^{n+1}}$$

and thus

$$Q^{-\frac{n}{2}}AQ^{-1}A^*Q^{-\frac{n}{2}} < \frac{n^n}{(n+1)^{n+1}}\, I,$$

$$AQ^{-1}A^* < \frac{n^n}{(n+1)^{n+1}}\, Q^n.$$

Since the function $\varphi(x) = x^n(1-x)$ is monotonically increasing on $(0, n/(n+1)]$ then for every number $\beta \in (0, n/(n+1)]$ we have $\beta^n(1-\beta) \leqslant \varphi_{\max}(x) = \varphi(n/(n+1)) = (n/(n+1))^n 1/(n+1)$. Thus there exists $\beta \in (0, n/(n+1)]$ for which

$$AQ^{-1}A^* \leqslant \beta^n(1-\beta)Q^n.$$

Moreover, if A is nonsingular, then there exists a number $\alpha \in (0, n/(n+1)]$ for which

$$\alpha^n(1-\alpha)Q^n \leqslant AQ^{-1}A^*.$$

Hence, if condition (4) holds, then there are two numbers $\alpha, \beta \in (0, n/(n+1)]$ and $\alpha < \beta$ for which

$$\alpha^n(1-\alpha)Q^n \leqslant A\,Q^{-1}\,A^* \leqslant \beta^n(1-\beta)Q^n. \tag{8}$$

Condition (4) is sufficiently for the existence a fixed point of $G(X)$ in K. Yet, this condition is not necessary. Further on, using property (8) we could consider the matrix sequence (7) with different initial points X_0. In the next section we consider these cases and the properties of the corresponding matrix sequences.

2.2. An iterative method

Consider the iterative method

$$X_{s+1} = G(X_s) = \sqrt[n]{A(Q-X_s)^{-1}A^*}, \quad s = 0, 1, 2, \ldots, \tag{9}$$

where X_0 is suitable chosen (see Theorems 11 and 13).

Theorem 9. *If Eq.* (1) *has a solution, then* $\{Z_s\}$ *is defined by* (9) *with* $Z_0 \in [0, \sqrt[n]{AQ^{-1}A^*}]$ *is monotonically increasing and converges to the smallest positive definite solution* X_S *of* (1).

Proof. We take Z_0 to be a positive definite matrix with $Z_0 \in [0, \sqrt[n]{AQ^{-1}A^*}]$. Let us consider the sequence $\{Y_s\}$ defined by (9) with $Y_0 = \sqrt[n]{AQ^{-1}A^*}$. According to Theorem 5 we have that this sequence converges to the minimal positive definite solution X_S. Consider the sequence $\{\hat{Z}_s\}$ defined by (9) with $\hat{Z}_0 = 0$.

We have

$$\hat{Z}_0 \leqslant Z_0 \leqslant Y_0$$

and then

$$\hat{Z}_1 = Y_0 \leqslant Z_1 \leqslant Y_1.$$

Assuming $\hat{Z}_s = Y_{s-1} \leqslant Z_s \leqslant Y_s$, it is easy to show that

$$\hat{Z}_{s+1} = Y_s \leqslant Z_{s+1} \leqslant Y_{s+1}. \tag{10}$$

But the sequence $\{Y_s\}$ converges to the minimal positive definite solution X_S of (1). We take limits in (10) and then conclude that the sequence $\{Y_s\}$ converges to the minimal positive definite solution X_S of (1). Hence every matrix sequence $\{Z_s\}$ with an initial point $Z_0 \in [0, \sqrt[n]{AQ^{-1}A^*}]$ converges to the minimal positive definite solution X_S of (1). $\square$

Theorem 10. *Let A be a nonsingular matrix. If matrix equation* (1) *has a positive definite solution X, then*

$$\sigma_m^2(Q^{-\frac{n}{2}}AQ^{-\frac{1}{2}}) \leqslant \frac{n^n}{(n+1)^{n+1}} \quad \textit{and} \quad X \geqslant \tilde{\alpha} Q,$$

where $\tilde{\alpha}$ is a solution of the nonlinear scalar equation $\alpha^n(1-\alpha)=\sigma_m^2(Q^{-\frac{n}{2}}AQ^{-\frac{1}{2}})$ and $\tilde{\alpha} \in (0, n/(n+1)]$.

Proof. We consider the sequence

$$\alpha_0 = 0, \quad \alpha_{s+1} = \sqrt[n]{\frac{\sigma_m^2(Q^{-\frac{n}{2}}AQ^{-\frac{1}{2}})}{1-\alpha_s}}, \quad s = 0, 1, 2, \ldots .$$

We will prove that for any positive definite solution X to (1) it is satisfied $X \geqslant \tilde{\alpha} Q$. Obviously, $X > \alpha_0 Q = 0$. Assuming that $X > \alpha_s Q$, then

$$X = \sqrt[n]{A(Q-X)^{-1}A^*} \geqslant \sqrt[n]{\frac{AQ^{-1}A^*}{1-\alpha_s}} \geqslant \sqrt[n]{\frac{\sigma_m^2(Q^{-\frac{n}{2}}AQ^{-\frac{1}{2}})Q^n}{1-\alpha_s}} = \alpha_{s+1} Q.$$

Hence $X > \alpha_s Q$ for every $s = 0, 1, \ldots$.

The number sequence $\{\alpha_s\}$ is monotonically increasing and thus the inequality $\alpha_s I < Q^{-\frac{1}{2}}XQ^{-\frac{1}{2}} < Q^{-\frac{1}{2}}QQ^{-\frac{1}{2}} < I$ leads that $\alpha_s < 1$ for every s. This sequence is bounded and hence it is convergent. Let $\tilde{\alpha}$ be its limit. Thus

$$\tilde{\alpha} = \sqrt[n]{\frac{\sigma_m^2(Q^{-\frac{n}{2}}AQ^{-\frac{1}{2}})}{1-\tilde{\alpha}}}$$

which means $\tilde{\alpha}$ is a solution of the following equation:

$$\alpha^n(1-\alpha) = \sigma_m^2(Q^{-\frac{n}{2}}AQ^{-\frac{1}{2}}).$$

But this equation has a solution if

$$\sigma_m^2(Q^{-\frac{n}{2}}AQ^{-\frac{1}{2}}) \leqslant \max_{x\in[0,1]} x^n(1-x) = \frac{n^n}{(n+1)^{n+1}}.$$

Since the maximum to the function $\varphi(x)$ is achieved for $x = n/(n+1) < 1$, then the equation has two solutions on $[0, 1]$. We will prove that $\tilde{\alpha} \in [0, n/(n+1)]$, or $\alpha_s < n/(n+1)$ for every s. Let $\alpha_s < n/(n+1)$ for some s. Then $1 - \alpha_s > 1/(n+1)$ and

$$\alpha_{s+1} = \sqrt[n]{\frac{\sigma_m^2(Q^{-\frac{n}{2}}AQ^{-\frac{1}{2}})}{1-\alpha_s}} < \sqrt[n]{\frac{n^n}{(n+1)^{n+1}}(n+1)} = \frac{n}{n+1}$$

and consequently $\tilde{\alpha} \in (0, n/(n+1)]$. $\square$

Theorem 11. *Let $\tilde{\alpha}$ and $\tilde{\beta}_1, \tilde{\beta}_2$ be solutions of scalar equations $\alpha^n(1-\alpha)=\sigma_m^2(Q^{-\frac{n}{2}}AQ^{-\frac{1}{2}})$ and $\beta^n(1-\beta)=\sigma_1^2(Q^{-\frac{n}{2}}AQ^{-\frac{1}{2}})$, respectively. Assume $0<\tilde{\alpha}\leqslant\tilde{\beta}_1\leqslant n/(n+1)\leqslant\tilde{\beta}_2<1$. Consider $\{X_s\}$ defined by* (9). *Then*

(i) *If $X_0=\gamma Q$ and $\gamma\in[0,\tilde{\alpha}]$, then $\{X_s\}$ is monotonically increasing and converges to the minimal positive definite solution X_{S} and $X_{\mathrm{S}}\in[\tilde{\alpha}\,Q,\tilde{\beta}_1\,Q]$.*
(ii) *If $X_0=\gamma Q$ and $\gamma\in[\tilde{\beta}_1,\tilde{\beta}_2]$, then $\{X_s\}$ is monotonically decreasing and converges to a positive definite solution $X_\gamma\in[\tilde{\alpha}Q,\gamma Q]$.*
(iii) *If $X_0=\gamma Q$ for $\gamma\in(\tilde{\alpha},\tilde{\beta}_1)$ and*

$$\min(q_1,q_2)<1,$$

where

$$q_1=\frac{1}{n}\left(\frac{1-\tilde{\alpha}}{\lambda_m(AQ^{-1}A^*)}\right)^{(n-1)/n}\left(\frac{1}{1-\tilde{\beta}_1}\,\|A\|\|Q^{-1}\|\right)^2$$

and

$$q_2=\frac{1}{n}\left(\frac{\lambda_1(AQ^{-1}A^*)}{1-\tilde{\beta}_1}\right)^{(n+1)/n}\|A^{-1}\|^2,$$

then $\{X_s\}$ converges to the minimal positive definite solution X_{S}.

Proof. Since the function $\varphi(x)$ is monotonically increasing where $x\in[0,n/(n+1)]$ we have $0<\alpha\leqslant\tilde{\alpha}\leqslant\tilde{\beta}_1\leqslant\beta\leqslant\tilde{\beta}_2\leqslant 1$ and the inequalities

$$\alpha^n(1-\alpha)I\leqslant Q^{-\frac{n}{2}}AQ^{-1}A^*Q^{-\frac{n}{2}}\leqslant\beta^n(1-\beta)I,$$
$$\alpha^n(1-\alpha)Q^n\leqslant AQ^{-1}A^*\leqslant\beta^n(1-\beta)Q^n$$

are satisfied.

(i) We have $X_0=\gamma\,Q\leqslant\tilde{\beta}_1 Q$ and $\gamma\in[0,\tilde{\alpha}]$. Thus

$$X_1=\sqrt[n]{A(Q-\gamma\,Q)^{-1}A^*}=\sqrt[n]{\frac{AQ^{-1}A^*}{1-\gamma}}\leqslant\sqrt[n]{\frac{1}{1-\gamma}\,\tilde{\beta}_1^n(1-\tilde{\beta}_1)Q^n}\leqslant\tilde{\beta}_1 Q$$

and

$$X_1=\sqrt[n]{\frac{AQ^{-1}A^*}{1-\gamma}}\geqslant\sqrt[n]{\frac{\tilde{\alpha}(1-\tilde{\alpha})}{1-\gamma}\,Q^n}\geqslant\gamma Q=X_0.$$

We have $X_0\leqslant X_1\leqslant\tilde{\beta}_1 Q$. We assume $X_{s-1}\leqslant X_s\leqslant\tilde{\beta}_1 Q$. Hence

$$(Q-X_{s-1})^{-1}\leqslant(Q-X_s)^{-1}\leqslant(Q-\tilde{\beta}_1 Q)^{-1}=\frac{1}{1-\tilde{\beta}_1}\,Q^{-1},$$

$$\sqrt[n]{A(Q-X_{s-1})^{-1}A^*} \leqslant \sqrt[n]{A(Q-X_s)^{-1}A^*} \leqslant \sqrt[n]{\frac{1}{1-\tilde{\beta}_1}\,AQ^{-1}A^*},$$

$$X_s \leqslant X_{s+1} \leqslant \sqrt[n]{\frac{1}{1-\tilde{\beta}_1}\,\tilde{\beta}_1^n(1-\tilde{\beta}_1)Q^n} = \tilde{\beta}_1 Q.$$

The sequence $\{X_s\}$ is monotonically increasing and converges to a positive definite solution $\tilde{X}$ with $\tilde{X} \leqslant \tilde{\beta}_1 Q$. We will show that for any positive definite solution X the inequality $X_s \leqslant X$ is true. For the $X_0 = \gamma Q$ we have $X_0 = \gamma Q \leqslant \tilde{\alpha} Q \leqslant X$. Assuming $X_{s-1} \leqslant X$ it is easy to show that $X_s \leqslant X$. So, the solution $\tilde{X}$ is the minimal positive definite solution X_S, i.e. $\tilde{X} \equiv X_S$ and according to Theorem 10 we conclude that $X_S \in [\tilde{\alpha} Q, \tilde{\beta}_1 Q]$.

(ii) Let $\gamma \in [\tilde{\beta}_1, \tilde{\beta}_2]$. Hence $Q \geqslant X_0 = \gamma Q \geqslant \tilde{\alpha} Q$. We have

$$X_1 \leqslant \sqrt[n]{\frac{1}{1-\gamma}\,\tilde{\beta}_1^n(1-\tilde{\beta}_1)\,Q^n} \leqslant \gamma Q = X_0$$

and

$$X_1 = \sqrt[n]{\frac{AQ^{-1}A^*}{1-\gamma}} \geqslant \sqrt[n]{\frac{1}{1-\gamma}\,\tilde{\alpha}^n(1-\tilde{\alpha})Q^n} \geqslant \tilde{\alpha} Q.$$

Thus $X_0 \geqslant X_1 \geqslant \tilde{\alpha} Q$. We assume $X_{s-1} \geqslant X_s \geqslant \tilde{\alpha} Q$. Then

$$\sqrt[n]{A(Q-X_{s-1})^{-1}A^*} \geqslant \sqrt[n]{A(Q-X_s)^{-1}A^*} \geqslant \sqrt[n]{\frac{AQ^{-1}A^*}{1-\tilde{\alpha}}},$$

$$X_s \geqslant X_{s+1} \geqslant \sqrt[n]{\frac{\tilde{\alpha}^n(1-\tilde{\alpha})Q^n}{1-\tilde{\alpha}}} = \tilde{\alpha} Q.$$

The sequence $\{X_s\}$ is monotonically decreasing and converges to a positive definite solution X_γ with the property $\gamma Q \geqslant X_\gamma \geqslant \tilde{\alpha} Q$.

(iii) Let $\gamma \in (\tilde{\alpha}, \tilde{\beta}_1)$ and the sequence $\{X_s\}$ is defined by (9). We will prove that $\{X_s\}$ is a Cauchy sequence. We have $\tilde{\beta}_1 Q > X_0 = \gamma Q > \tilde{\alpha} Q$. Assume $\tilde{\alpha} Q < X_s < \tilde{\beta}_1 Q$. For X_{s+1} we compute

$$X_{s+1} = \sqrt[n]{A(Q-X_s)^{-1}A^*} < \sqrt[n]{\frac{AQ^{-1}A^*}{1-\tilde{\beta}_1}} \leqslant \tilde{\beta}_1 Q,$$

$$X_{s+1} > \sqrt[n]{\frac{AQ^{-1}A^*}{1-\tilde{\alpha}}} \geqslant \tilde{\alpha} Q.$$

So, $\tilde{\alpha} Q < X_s < \tilde{\beta}_1 Q$ for $s = 0, 1, \ldots$. Let us consider the difference $X_{s+p} - X_s$ for which we obtain

$$X_{s+p} - X_s = \sqrt[n]{A(Q-X_{s+p-1})^{-1}A^*} - \sqrt[n]{A(Q-X_{s-1})^{-1}A^*}.$$

Since $\tilde{\beta}_1 Q > X_j > \tilde{\alpha} Q$ for $j = 0, 1, \ldots$ we have

$$A(Q - X_j)^{-1}A^* > \frac{1}{1-\tilde{\alpha}}\, AQ^{-1}A^* \geqslant \frac{\lambda_m(AQ^{-1}A^*)}{1-\tilde{\alpha}}\, I,$$

$$(Q - X_j)^{-1} < \frac{1}{1-\tilde{\beta}_1}\, Q^{-1} \leqslant \frac{\lambda_1(Q^{-1})}{1-\tilde{\beta}_1}\, I.$$

Hence

$$A(Q - X_{s+p-1})^{-1}A^* > \frac{\lambda_m(AQ^{-1}A^*)}{1-\tilde{\alpha}}\, I \quad \text{and} \quad A(Q - X_{s-1})^{-1}A^* > \frac{\lambda_m(AQ^{-1}A^*)}{1-\tilde{\alpha}}\, I.$$

Using Theorem 1 we have

$$\|X_{s+p} - X_s\| \leqslant \frac{1}{n}\, \zeta^{\frac{1-n}{n}}\, \|A[(Q - X_{s+p-1})^{-1} - (Q - X_{s-1})^{-1}]A^*\|,$$

where $\zeta = \frac{\lambda_m(AQ^{-1}A^*)}{1-\tilde{\alpha}}$. Thus

$$\begin{aligned}
\|X_{s+p} - X_s\| &\leqslant \frac{1}{n}\, \zeta^{\frac{1-n}{n}}\, \|A\|^2 \|(Q - X_{s-1})^{-1}(X_{s-1} - X_{s+p-1})(Q - X_{s+p-1})^{-1}\| \\
&\leqslant \frac{1}{n}\, \zeta^{\frac{1-n}{n}}\, \|A\|^2 \left(\frac{1}{1-\tilde{\beta}_1}\, \|Q^{-1}\|\right)^2 \|X_{s+p-1} - X_{s-1}\| \\
&\leqslant \cdots \\
&\leqslant \left(\frac{1}{n}\, \zeta^{\frac{1-n}{n}} \left(\frac{1}{1-\tilde{\beta}_1}\, \|A\| \|Q^{-1}\|\right)^2\right)^s \|X_p - X_0\| \\
&= q_1^s \|X_p - X_0\|.
\end{aligned}$$

Follow the proof of Theorem 2 with $\beta = \tilde{\beta}$ we have

$$\begin{aligned}
\|X_{s+p} - X_s\|_F &\leqslant \frac{1}{n} \left(\frac{\lambda_1(AQ^{-1}A^*)}{1-\tilde{\beta}_1}\right)^{(n+1)/n} \|A^{-1}\|^2 \|X_{s+p-1} - X_{s-1}\|_F \\
&\leqslant \cdots \\
&= q_2^s \|X_p - X_0\|_F.
\end{aligned}$$

Since $\min(q_1, q_2) < 1$ we obtain

$$\|X_{s+p} - X_s\| \leqslant \frac{q_1^s}{1-q_1}\, \|X_1 - X_0\|$$

or

$$\|X_{s+p} - X_s\|_F \leqslant \frac{q_2^s}{1-q_2}\, \|X_1 - X_0\|_F$$

for all $s,\ p = 0, 1 \ldots .$

The sequence $\{X_s\}$ is a Cauchy sequence, considered in the Banach space $[\tilde{\alpha}Q, \tilde{\beta}_1 Q]$. Hence this sequence has a positive definite limit $\tilde{X}$, which is a unique positive definite solution of (1) on $[\tilde{\alpha}Q, \tilde{\beta}_1 Q]$. According to case (i), Eq. (1) has the minimal positive definite solution $X_S \geqslant \tilde{\alpha}\, Q$. Thus $\tilde{X} \equiv X_S$. $\square$

Remark. If (4) holds, then (8) is true. Hence, the $G(X)$ has a fixed point and according to Theorem 11 it follows that X_S and X_γ are fixed points of $G(X)$ in K. Under additional restriction on the matrix Q we are proving the following corollary.

Corollary 12. *Let Q be unitary and there exists a number $\beta \in (0, n/(n+1))$ with property $AQ^{-1}A^* \leqslant \beta^n(1-\beta)Q^n$. Then there exists a fixed point of $G(X)$ in K.*

Proof. Using the inequality $AQ^{-1}A^* \leqslant \beta^n(1-\beta)Q^n$ for $\beta \leqslant n/(n+1)$ we have

$$AQ^{-1}A^* \leqslant \left(\frac{n}{n+1}\right)^n \frac{1}{n+1}\, Q^n$$

and thus

$$\|AQ^{-\frac{1}{2}}\|^2\, \|Q^{-1}\|^n = \|AQ^{-1}A^*\| \leqslant \left(\frac{n}{n+1}\right)^n \frac{1}{n+1}\, \|Q\|^n = \frac{n^n}{(n+1)^{n+1}},$$

because $\|Q\| = \|Q^*\| = \|Q^{-1}\| = 1$. $\square$

In order to use Theorem 11, it is necessary to compute the matrices Q^n, $Q^{-\frac{n}{2}}$ and to find the singular values of the matrix $Q^{-\frac{n}{2}}AQ^{-\frac{1}{2}}$. Now, we shall prove a new theorem under an additional restriction on the matrix Q, where it is enough to use the matrix $AQ^{-\frac{1}{2}}$ and its singular values.

Theorem 13. *Let $Q \geqslant I$ and $\tilde{\alpha}$ and $\tilde{\beta}_1, \tilde{\beta}_2$ be solutions of scalar equations $\alpha^n(1-\alpha) = \sigma_m^2(AQ^{-\frac{1}{2}})$ and $\beta^n(1-\beta) = \sigma_1^2(AQ^{-\frac{1}{2}})$, respectively. Denote $\tilde{\tau} = \sqrt[n]{\sigma_m^2(AQ^{-\frac{1}{2}})}$. Assume $0 < \tilde{\tau} < \tilde{\alpha} \leqslant \tilde{\beta}_1 \leqslant n/(n+1) \leqslant \tilde{\beta}_2 < 1$. Consider $\{X_s\}$ defined by (9) with suitable chosen X_0. Then*

(i) *If $X_0 = \gamma I$ and $\gamma \in [0, \tilde{\tau}]$, then $\{X_s\}$ is monotonically increasing and converges to X_S.*
(ii) *If $X_0 = \gamma Q$ and $\gamma \in [\tilde{\beta}_1, \tilde{\beta}_2]$, then $\{X_s\}$ is monotonically decreasing and converges to a positive definite solution X_γ and $X_\gamma \in (\tilde{\alpha}\sqrt[n]{1-\tilde{\alpha}}I, Q)$.*
(iii) *If $X_0 = \gamma Q$ and $\gamma \in (\tilde{\tau}, \tilde{\beta}_1)$, and*

$$\min(q_1, q_2) < 1,$$

where

$$q_1 = \frac{1}{n}\left(\frac{1}{\lambda_m(AQ^{-1}A^*)}\right)^{(n-1)/n} \left(\frac{1}{1-\tilde{\beta}_1}\, \|A\|\|Q^{-1}\|\right)^2$$

and

$$q_2 = \frac{1}{n}\left(\frac{\lambda_1(AQ^{-1}A^*)}{1-\tilde{\beta}_1}\right)^{(n+1)/n} \|A^{-1}\|^2,$$

then $\{X_s\}$ converges to the minimal positive definite solution X_S.

Proof. The function $\varphi(x)$ is monotonically increasing in $[0, n/(n+1)]$. we have $0 < \alpha \leqslant \tilde{\alpha} \leqslant \tilde{\beta}_1 \leqslant \beta \leqslant \tilde{\beta}_2 \leqslant 1$ and the inequalities

$$\alpha^n(1-\alpha)I \leqslant AQ^{-1}A^* \leqslant \beta^n(1-\beta)I$$

hold.

(i) Let $X_0 = \gamma I$ and $\gamma \in [0, \tilde{\tau}]$. We have $\gamma \leqslant \sqrt[n]{\sigma_m^2(AQ^{-\frac{1}{2}})} = \sqrt[n]{\tilde{\alpha}^n(1-\tilde{\alpha})} \leqslant \sqrt[n]{\tilde{\alpha}^n} \leqslant \tilde{\beta}_1$ and $0 \leqslant X_0 = \gamma I \leqslant \gamma Q \leqslant \tilde{\beta}_1 Q \leqslant Q$. Thus

$$X_1 \leqslant \sqrt[n]{\frac{1}{1-\gamma}\tilde{\beta}_1^n(1-\tilde{\beta}_1)I} \leqslant \tilde{\beta}_1 I \leqslant \tilde{\beta}_1 Q$$

and

$$X_1 = \sqrt[n]{AQ^{-\frac{1}{2}}(I-\gamma Q^{-1})^{-1}Q^{-\frac{1}{2}}A^*}.$$

Since $Q^{-1} \leqslant I$ we have $\|\gamma Q^{-1}\| = \gamma\|Q^{-1}\| < 1$ and then

$$(I-\gamma Q^{-1})^{-1} = I + \gamma Q^{-1} + \cdots > I$$

$$X_1 > \sqrt[n]{AQ^{-1}A^*} \geqslant \sqrt[n]{\tilde{\alpha}^n(1-\tilde{\alpha})}I = \sqrt[n]{\sigma_m^2(AQ^{-\frac{1}{2}})}I = \tilde{\tau}I \geqslant \gamma I = X_0.$$

Thus

$$\sqrt[n]{AQ^{-1}A^*} \geqslant \tilde{\tau}I. \tag{11}$$

Further on, assume $X_{s-1} \leqslant X_s \leqslant \tilde{\beta}_1 Q$, we have

$$\sqrt[n]{A(Q-X_{s-1})^{-1}A^*} \leqslant \sqrt[n]{A(Q-X_s)^{-1}A^*} \leqslant \sqrt[n]{\frac{1}{1-\tilde{\beta}}AQ^{-1}A^*} \leqslant \tilde{\beta}Q$$

$$X_s \leqslant X_{s+1} \leqslant \tilde{\beta}Q.$$

The sequence $\{X_s\}$ is monotonically increasing and converges to a positive definite solution $\tilde{X}$ with $\tilde{X} \leqslant \tilde{\beta}_1 Q$. Thus combine (11) and Theorem 9 we have $\tilde{X} \equiv X_S$.

(ii) Let $X_0 = \gamma Q$ and $\gamma \in [\tilde{\beta}_1, \tilde{\beta}_2]$. We will prove that the sequence $\{X_s\}$ is bounded by below to $\tilde{\tau} I$. We have $Q \geqslant X_0 = \gamma Q > \gamma I > \tilde{\tau} I$. For X_1 we obtain

$$X_1 \leqslant \sqrt[n]{\frac{1}{1-\gamma}\tilde{\beta}_1^n(1-\tilde{\beta}_1)I} \leqslant \gamma I \leqslant \gamma Q = X_0$$

and

$$X_1 = \sqrt[n]{\frac{1}{1-\gamma}AQ^{-1}A^*} \geqslant \sqrt[n]{\frac{1}{1-\gamma}\tilde{\alpha}^n(1-\tilde{\alpha})I} \geqslant \sqrt[n]{\frac{\tilde{\tau}^n}{1-\gamma}I} \geqslant \tilde{\tau}I.$$

Hence $\tilde{\tau} I < X_1 < X_0$. We assume $\tilde{\tau} I < X_s < X_{s-1}$ and then

$$\sqrt[n]{AQ^{-\frac{1}{2}}(I - \tilde{\alpha}Q^{-1})^{-1}Q^{-\frac{1}{2}}A^*} < X_{s+1} < X_s.$$

Since $\|\tilde{\tau}\, Q^{-1}\| \leqslant \tilde{\tau}\|Q^{-1}\| < 1$, we have $(I - \tilde{\tau}\, Q^{-1})^{-1} > I$ and

$$\sqrt[n]{AQ^{-\frac{1}{2}}(I - \tilde{\tau}\, Q^{-1})^{-1}Q^{-\frac{1}{2}}A^*} > \sqrt[n]{AQ^{-1}A^*} \geqslant \sqrt[n]{\tilde{\alpha}^n(1-\tilde{\alpha})}I = \tilde{\tau} I.$$

Thus $\tilde{\tau} I < X_{s+1} < X_s$. The sequence $\{X_s\}$ is monotonically decreasing and converges to a positive definite solution X_γ with the property $\gamma\, Q \geqslant X_\gamma \geqslant \tilde{\tau}\, I$.

(iii) Let $\gamma \in (\tilde{\tau}, \tilde{\beta}_1)$ and the sequence $\{X_s\}$ is defined by (9). Combine (i) and (ii) we have $\tilde{\beta}_1\, Q > X_j > \tilde{\tau} I$ for $j = 0, 1, \ldots$ and hence

$$A(Q - X_j)^{-1}A^* > A(Q - \tilde{\tau} I)^{-1}A^* > AQ^{-1}A^* \geqslant \lambda_m(AQ^{-1}A^*)I,$$

$$(Q - X_j)^{-1} < \frac{1}{1-\tilde{\beta}_1}\, Q^{-1} \leqslant \frac{\lambda_1(Q^{-1})}{1-\tilde{\beta}_1}\, I.$$

Consider the difference $X_{s+p} - X_s$. Using Theorem 1, analogous of item (iv), Theorem 11 we have

$$\begin{aligned}\|X_{s+p} - X_s\| &\leqslant \left(\frac{1}{n}\,(\lambda_m(AQ^{-1}A^*))^{(1-n)/n}\left(\frac{1}{1-\tilde{\beta}_1}\,\|A\|\|Q^{-1}\|\right)^2\right)^s \|X_p - X_0\| \\ &= q_1^s\|X_p - X_0\|\end{aligned}$$

and

$$\|X_{s+p} - X_s\|_F \leqslant q_2^s\|X_p - X_0\|_F.$$

Since $\min(q_1, q_2) < 1$ we obtain

$$\|X_{s+p} - X_s\| \leqslant \frac{q_1^s}{1-q_1}\,\|X_1 - X_0\|$$

or

$$\|X_{s+p} - X_s\|_F \leqslant \frac{q_2^s}{1-q_2}\,\|X_1 - X_0\|_F$$

for all $s, p = 0, 1 \ldots$.

Thus, the sequence $\{X_s\}$ is a Cauchy sequence in $[\tilde{\tau} I, \tilde{\beta}_1\, Q]$. Analogously, we obtain that the limit of this sequence is the minimal positive definite solution X_S of (1). □

In Theorem 13 we compute the singular values of $AQ^{-\frac{1}{2}}$ instead of the singular values of $Q^{-\frac{n}{2}}AQ^{-\frac{1}{2}}$ as in Theorem 11. Yet, the convergence rate q_1 obtained in Theorem 13 is bigger that the corresponding obtained in Theorem 11.

Now, we are proving the following theorem:

Theorem 14. *Let α_1 and β_1 by real for which the inequalities*

(i) $\alpha_1^n(1-\alpha_1)Q^n \leqslant AQ^{-1}A^* \leqslant \beta_1^n(1-\beta_1)Q^n$.
(ii) $\min(q_1, q_2) < 1$, *where*

$$q_1 = \frac{1}{n}\left(\frac{1-\alpha_1}{\lambda_m(AQ^{-1}A^*)}\right)^{(n-1)/n} \|A\|^2 \left(\frac{\|Q^{-1}\|}{1-\beta_1}\right)^2$$

and

$$q_2 = \frac{1}{n}\left(\frac{\lambda_1(AQ^{-1}A^*)}{1-\beta_1}\right)^{(n+1)/n} \|A^{-1}\|^2$$

hold. Then Eq. (1) *has a unique solution on* $(0, \beta_1 Q)$ *which is the minimal solution* X_S.

Proof. According to condition (i) there exist reals γ_1 and γ_2 for which

$$0 \leqslant \alpha_1 \leqslant \gamma_1 \leqslant \tilde{\alpha} \leqslant \tilde{\beta}_1 \leqslant \gamma_2 \leqslant \beta_1 \leqslant 1,$$

where $\tilde{\alpha}$ and $\tilde{\beta}_1 \leqslant n/(n+1)$ are solutions of the equations $\alpha^n(1-\alpha) = \sigma_m^2(AQ^{-\frac{1}{2}})$ and $\beta^n(1-\beta) = \sigma_1^2(AQ^{-\frac{1}{2}})$, respectively.

The recurrence Eq. (9) defines two matrix sequences $\{X_k'\}$ and $\{X_k''\}$ with initial points $X_0' = \gamma_1 Q$ and $X_0'' = \gamma_2 Q$. Now, we will prove that these sequences are convergent to the same limit $\tilde{X}$ which is a positive definite solution of (1).

We have $\alpha_1 Q \leqslant X_s' \leqslant \beta_1 Q$ and $\alpha_1 Q \leqslant X_s'' \leqslant \beta_1 Q$ for all $s = 0, 1, \ldots$.

Using the same approach as in Theorem 11 we obtain

$$\|X_s' - X_s''\| \leqslant \frac{1}{n}\left(\frac{\lambda_m(AQ^{-1}A^*)}{1-\alpha_1}\right)^{(1-n)/n} \|A\|^2 \left(\frac{\|Q^{-1}\|}{1-\beta_1}\right)^2 \|X_{s-1}' - X_{s-1}''\|.$$

According to Theorem 2 with $\beta = \beta_1$ we have

$$\|X_s' - X_s''\|_F \leqslant \frac{1}{n}\left(\frac{\lambda_1(AQ^{-1}A^*)}{1-\beta_1}\right)^{(n+1)/n} \|A^{-1}\|^2 \|X_{s-1}' - X_{s-1}''\|_F.$$

Thus

$$\|X_s' - X_s''\| < \|X_{s-1}' - X_{s-1}''\|$$

or

$$\|X_s' - X_s''\|_F < \|X_{s-1}' - X_{s-1}''\|_F.$$

Consequently sequences $\{X_s'\}$ and $\{X_s''\}$ have the common limit $\tilde{X}$ which is a unique positive definite solution of (1). The solution $\tilde{X}$ which is a positive definite solution of (1) and $\tilde{X}$ is a unique on $(\alpha_1 Q, \beta_1 Q)$. In Theorem 11 (i) we have proved that the matrix sequence $\{X_s\}$ with $X_0 \leqslant \tilde{\alpha}Q$ converges to X_S. Thus

Table 1
Iterative method (9) with different initial points, $tol = 1.0e - 14$

m	Initial point	Number of iterations	Error
4	$X_0 = \tilde{\alpha}Q$	$K_{X_S} = 242$	$\varepsilon(X_S) = 7.9381e - 015$
	$X_0 = \tilde{\beta}_1 Q$	$K_{X_{\tilde{\beta}_1}} = 240$	$\varepsilon(X_{\tilde{\beta}_1}) = 6.9944e - 015$

$X_S \equiv \tilde{X}$. So, under conditions of this theorem the matrix sequence $\{X_s\}$ with $X_0 \leqslant \beta_1 Q$ converges to X_S which is a unique solution of (1) on $(0, \beta_1 Q)$ and $X_S \in (\tilde{\alpha}Q, \tilde{\beta}_1 Q)$. The theorem is proved. □

2.3. Numerical experiments

We have made numerical experiments to compute a positive definite solution of the equation $X + A^* X^{-n} A = Q$. Computations were done on a PENTIUM IV, 2.1 GHz computer. All programs were written in MATLAB. We denote $\varepsilon(Z) = \|Z + A^* Z^{-n} A - Q\|_\infty$.

We have tested our iteration processes for solving the equation $X + A^* X^{-2} A = Q$ on the following $m \times m$ matrices. We use the stopping criterion $\varepsilon(Z) < tol$, ($tol = 1.0e - 14$) and let K_{X_0} be the smallest number s, for which $\varepsilon(X_s) < tol$ for method (9) with an initial point X_0. We use the reals $\tilde{\alpha} < \tilde{\beta}_1 \leqslant n/(n+1)$ which are solutions of the equations $\alpha^n(1-\alpha) = \sigma_m^2(Q^{-\frac{n}{2}} A Q^{-\frac{1}{2}})$ and $\beta^n(1-\beta) = \sigma_1^2(Q^{-\frac{n}{2}} A Q^{-\frac{1}{2}})$, respectively.

Example. Consider the equation $X + A^* X^{-2} A = Q$ and elements a_{ij} of the matrix A are computed by

$$a_{ij} = \begin{cases} \dfrac{i+j-m}{200} & \text{if } i < j, \\ \dfrac{j-i-m}{200} & \text{if } i > j, \\ \dfrac{i+j+m}{n^2} & \text{if } i = j. \end{cases}$$

We define the $m \times m$ matrix Q as follows:

$$Q = U^{\mathrm{T}} diag\left[1 + \frac{\vartheta.1}{m}; 1 + \frac{\vartheta.2}{m}; \ldots, 1 + \frac{\vartheta.m}{m}\right] U,$$

where $\vartheta = 10.5$, $U = I - 2v^{\mathrm{T}} v$ and $v = (\frac{1}{\sqrt{m}}, \ldots, \frac{1}{\sqrt{m}})$.

Note that, for this example condition (4) is not satisfied, while condition (8) is satisfied for $\tilde{\alpha} = 0.0671$ and $\tilde{\beta}_1 = 0.6508$. The factors p and q defined in Theorem 6 are bigger than 1 ($p = 1.4791$ and $q = 1.4231$) and the factors q_1 and q_2 defined in Theorem 11 are bigger than 1 ($q_1 = 3.7657$ and $q_2 = 1.3794$), too. We use iterative method (9) for computing a positive definite solution with different initial points. The results are given in next Table 1. According to Theorem 11 we have that matrix sequences (9) with $X_0 = \tilde{\alpha}Q$ converges to X_S and the same matrix sequence with $X_0 = \tilde{\beta}_1 Q$ converges to $X_{\tilde{\beta}_1}$. Numerical experiments show that $X_S \equiv X_{\tilde{\beta}_1}$ for this example.

Table 2
Recurrence Eq. (9) with different initial points, $\vartheta = 10.9$, $tol = 1.0e - 14$

m	Initial point	Number of iteration	Error
4	$X_0 = 0.0537Q$	$K_{X_S} = 68$	$\varepsilon(X_S) = 8.7708e - 015$
	$X_0 = \tilde{\alpha}Q$	$K_{X_S} = 67$	$\varepsilon(X_S) = 9.4091e - 015$
	$X_0 = \tilde{\beta}_1 Q$	$K_{X_S} = 58$	$\varepsilon(X_S) = 9.1871e - 015$
	$X_0 = \frac{\tilde{\alpha} + \tilde{\beta}_1}{2} Q$	$K_{X_S} = 63$	$\varepsilon(X_S) = 9.4924e - 015$

Further on, we consider the same example where $\vartheta = 10.9$. For this new matrix Q condition (4) holds and thus there exists a fixed point for $G(X)$ in K. Yet, the factors p and q (Theorem 6) are bigger that 1 ($p = 1.4191$ and $q = 1.4821$). Hence, we could not say whether $G(X)$ has a unique fixed point in K. Besides, there exists reals $\tilde{\alpha} = 0.0637$ and $\tilde{\beta}_1 = 0.5509$, for which inequalities (8) hold. The sequence $\{X_s\}$ for $X_0 = \tilde{\alpha}Q$ is monotonically increasing and the same sequence with $X_0 = \tilde{\beta}_1 Q$ is monotonically decreasing. The factors q_1 and q_2 defined in Theorem 11 are $q_1 = 2.1972$ and $q_2 = 0.9075$. Since $q_2 < 1$, then the sequence $\{X_s\}$ with $X_0 = \gamma Q$, $\gamma = \frac{1}{2}(\tilde{\alpha} + \tilde{\beta}_1)$ converges to the minimal positive definite solution X_S. Moreover, we find reals $\alpha_1 = 0.0537 < \tilde{\alpha}$ $\beta_1 = \tilde{\beta}_1$ for which condition (i) of Theorem 14 holds and the corresponding factors are $q_1 = 2.2089$ and $q_2 = 0.9075 < 1$. Thus the sequence $\{X_s\}$ with any $X_0 \in (0, \beta_1 Q)$ converges to X_S. The results with different initial points are given in next Table 2.

2.4. *The special solution X_1*

In this section we investigate the special positive definite solution X_1 of (1).

Now, we consider the recurrence equation

$$X_0 = \gamma Q, \quad X_{s+1} = Q - A^* X_s^{-n} A, \tag{12}$$

where $\gamma \in (n/(n+1), 1)$ and study the corresponding matrix sequences constructed after using different starting points.

It is well known some properties of this solution.

Theorem 15 (*Theorem 1.1.4, Hasanov [4]*). *If* $\|A\|\sqrt{\|Q^{-1}\|^{n+1}} < \sqrt{\frac{n^n}{(n+1)^{n+1}}}$, *then the matrix sequence* $\{X_s\}$ *defined by* (12) *with* $X_0 = \eta I$, $\eta \in (\frac{n}{(n+1)\|Q^{-1}\|}, \frac{1}{\|Q^{-1}\|}]$ *converges to a positive definite solution* X_1 *of* (1) *and the inequalities*

$$\frac{n}{(n+1)\|Q^{-1}\|} I < X_1 \leqslant Q - \frac{1}{(\|Q\|\,\|Q^{-1}\|)^{n-1}} A^* Q^{-n} A, \tag{13}$$

hold. Moreover, the solution X_1 is unique with these properties.

Theorem 16 (*Corollary 1.1.5, Hasanov [4]*). *If* $\|A\|\sqrt{\|Q^{-1}\|^{n+1}} < \sqrt{\frac{n^n}{(n+1)^{n+1}}}$, *then the solution* X_1 *of* (1) *satisfies the inequalities*

(i) $\|X_1^{-1}\| < \frac{n+1}{n}\|Q^{-1}\|$.
(ii) $\frac{n+1}{n}\|Q\| < \|X_1\|$.

It is easy to show that if there exists a largest positive definite solution X_{L} of (1), then $X_1 = X_{\mathrm{L}}$. We define the set of Hermitian positive definite matrices

$$W = \left\{ Y \in \mathscr{P}(m) : \|Y\| \leqslant \frac{n+1}{n}\|Q^{-1}\| \right\}$$

and consider the matrix function

$$Y = F(Y) = (Q - A^* Y^n A)^{-1}.$$

We will prove that if there exists a unique fixed point of $F(Y)$ in W, then this fixed point is the inverse matrix of X_1.

Now, Eq. (1) can be rewrite of the form $Y = F(Y)$ where $Y = X^{-1}$. It is obvious that if X is a solution of (1), then $Y = X^{-1}$ is a solution of the equation $Y = F(Y)$ and conversely.

Theorem 17. *If*

$$\|A\|^2 \|Q^{-1}\|^{n+1} < \frac{n^n}{(n+1)^{n+1}}, \tag{14}$$

then there exists a unique fixed point for $F(Y)$ *in W. Moreover, this fixed point is a inverse matrix of the special positive definite* X_1.

Proof. We will prove that the matrix function $F(Y)$ defines continuous map on the set of Hermitian positive definite matrices W.

It is obvious that W is a convex, closed and bounded set. Since $\|Q^{-\frac{1}{2}} A^* Y^n A Q^{-\frac{1}{2}}\| \leqslant \|Q^{-1}\| \|A\|^2 \|Y\|^n \leqslant 1/(n+1)$ for every $Y \in W$ we obtain that $F(Y)$ is a continuous map in W. Besides

$$\begin{aligned} \|F(Y)\| &\leqslant \|(Q - A^* Y^n A)^{-1}\| \\ &\leqslant \frac{\|Q^{-1}\|}{1 - \|Q^{-\frac{1}{2}} A^* Y^n A Q^{-\frac{1}{2}}\|} \\ &\leqslant \frac{\|Q^{-1}\|}{1 - \frac{1}{n+1}} = \frac{n+1}{n}\|Q^{-1}\| \end{aligned}$$

we conclude that $F(Y) \in W$. Hence $F(Y)$ has a fixed point on W. We will prove that $F(Y)$ is a contraction in W. For every $Y \in W$ and $Z \in W$ consider the $\|F(Y) - F(Z)\|$. We have

$$
\begin{aligned}
\|F(Y) - F(Z)\| &= F(Y)(F(Z)^{-1} - F(Y)^{-1})F(Z) \\
&\leqslant \|F(Y)\|\,\|F(Z)\|\,\|F(Z)^{-1} - F(Y)^{-1}\| \\
&\leqslant \left(\frac{n+1}{n}\,\|Q^{-1}\|\right)^2 \|A\|^2\,\|Y^n - Z^n\| \\
&\leqslant \left(\frac{n+1}{n}\,\|Q^{-1}\|\right)^2 \|A\|^2 \sum_{j=1}^{n} \|Y\|^{n-j}\,\|Y - Z\|\,\|Z\|^{j-1} \\
&\leqslant \left(\frac{n+1}{n}\,\|Q^{-1}\|\right)^2 \|A\|^2\, n \left(\frac{n+1}{n}\,\|Q^{-1}\|\right)^{n-1} \|Y - Z\| \\
&= \frac{(n+1)^{n+1}}{n^n}\,\|Q^{-1}\|^{n+1}\,\|A\|^2\,\|Y - Z\|.
\end{aligned}
$$

From (14) it follows that

$$
\frac{(n+1)^{n+1}}{n^n}\,\|Q^{-1}\|^{n+1}\|A\|^2 < 1
$$

and thus $F(Y)$ is a contraction. Hence, the map $F(Y)$ has a unique fixed point $\hat{Y}$ in W, which is a Hermitian positive definite matrix. This fixed point is a limit of the matrix sequence $Y_{k+1} = F(Y_k)$ for every $Y_0 \in W$. Thus $\hat{Y}^{-1}$ is the unique positive definite solution of (1) with the property $\|\hat{Y}\| \leqslant (n+1)/n\,\|Q^{-1}\|$, i.e. $\hat{Y}^{-1} \equiv X_1$. □

Theorem 18. *If matrix equation* (1) *with nonsingular matrix A has a positive definite solution X, then $X \leqslant \xi Q$, where ξ is a solution to $\alpha^n(1-\alpha) = [1/v_Q]\,\sigma_m^2(Q^{-\frac{n}{2}} A Q^{-\frac{1}{2}})$ on $[n/(n+1), 1]$ and $v_Q = (M_Q/m_Q)^{n-1}$, $m_Q I \leqslant Q \leqslant M_Q I$.*

Proof. Consider the sequence

$$
\alpha_0 = 1, \quad \alpha_{s+1} = 1 - \frac{\sigma_m^2(Q^{-\frac{n}{2}} A Q^{-\frac{1}{2}})}{\alpha_s^n v_Q}.
$$

We will prove that for any positive definite solution X the inequality $X \leqslant \xi Q$ is satisfied. Obviously, $X \leqslant \alpha_0 Q = Q$. Assuming $X \leqslant \alpha_s Q$ and according to Theorem 2, we obtain $X^n \leqslant \alpha_s^n v_Q Q^n$ and

$$
\begin{aligned}
X = Q - A^* X^{-n} A \leqslant Q - \frac{A^* Q^{-n} A}{\alpha_s^n v_Q} &= Q^{\frac{1}{2}} \left(I - \frac{Q^{-\frac{1}{2}} A^* Q^{-n} A Q^{-\frac{1}{2}}}{\alpha_s^n v_Q} \right) Q^{\frac{1}{2}} \\
&\leqslant Q^{\frac{1}{2}} \left(1 - \frac{\sigma_m^2\left(Q^{-\frac{n}{2}} A Q^{-\frac{1}{2}}\right)}{\alpha_s^n v_Q} \right) Q^{\frac{1}{2}} = \tilde{\alpha}_{s+1} Q.
\end{aligned}
$$

Thus $X \leqslant \alpha_s Q$ for every $s = 0, 1, \ldots$.

The sequence $\{\alpha_s\}$ is monotonically decreasing and bounded and hence it is convergent. Let ξ be its limit. Thus

$$\xi = 1 - \frac{\sigma_m^2(Q^{-\frac{n}{2}}AQ^{-\frac{1}{2}})}{\xi^n v_Q}$$

which means that ξ is a solution to the equation

$$\xi^n(1-\xi) = \frac{1}{v_Q}\sigma_m^2(Q^{-\frac{n}{2}}AQ^{-\frac{1}{2}}). \tag{15}$$

Yet, in order for the last equation to have a solution the next inequality must be satisfied

$$\frac{1}{v_Q}\sigma_m^2(Q^{-\frac{n}{2}}AQ^{-\frac{1}{2}}) \leqslant \max_{x\in[0,1]} \varphi(x) = \frac{n^n}{(n+1)^{n+1}}. \tag{16}$$

According to Theorem 10 we have

$$\sigma_m^2(Q^{-\frac{n}{2}}AQ^{-\frac{1}{2}}) \leqslant \frac{n^n}{(n+1)^{n+1}},$$

whence the inequality (16) holds. Eq. (15) has two solutions on $[0, 1]$. We will show that $\xi \in [n/(n+1), 1]$ or $n/(n+1) < \alpha_s < 1$ for every s. Let for some s we have $n/(n+1) < \alpha_s < 1$. We compute

$$\frac{\sigma_m^2(Q^{-\frac{n}{2}}AQ^{-\frac{1}{2}})}{\alpha_s^n}\frac{1}{v_Q} < \frac{n^n}{(n+1)^{n+1}}\frac{(n+1)^n}{n^n} = \frac{1}{n+1}.$$

Thus

$$\alpha_{s+1} > 1 - \frac{1}{n+1} = \frac{n}{n+1}$$

and hence $\xi \in [n/(n+1), 1]$. $\square$

Theorem 19. *If there exist numbers α, β such that*

(i) $\frac{n}{n+1} < \alpha \leqslant \beta \leqslant 1$.
(ii) $v_Q\beta^n(1-\beta)Q \leqslant A^*Q^{-n}A \leqslant \frac{1}{v_Q}\alpha^n(1-\alpha)Q$,
where $v_Q = (\frac{M_Q}{m_Q})^{n-1}$, $m_QI \leqslant Q \leqslant M_QI$.
(iii) $q = \frac{n\|A\|^2\|Q^{-1}\|^{n+1}}{\alpha^{n+1}} < 1$.

Then iterative process (12) with $\alpha \leqslant \gamma \leqslant \beta$ converges to a unique positive definite solutions $\tilde{X}$ of $X + A^*X^{-n}A = Q$ and $[n/(n+1)]Q \leqslant \tilde{X} \leqslant Q$.

Proof. Note that if β does not exist we can take $\beta = 1$.

We shall show that the matrix sequences $\{X_s\}$ is a Cauchy sequence and for each s we have $\alpha Q \leqslant X_s \leqslant \beta Q$. Suppose $X_0 = \gamma Q$, $(\alpha \leqslant \gamma \leqslant \beta)$. Hence $\alpha Q \leqslant X_0 \leqslant \beta Q$. We get

$$X_1 = Q - \frac{1}{\gamma^n} A^* Q^{-n} A \leqslant Q - \frac{1}{\gamma^n} v_Q \beta^n (1-\beta) Q,$$

$$X_1 \leqslant Q(1-(1-\beta)) = \beta Q.$$

Similarly, we have

$$X_1 = Q - \frac{1}{\gamma^n} A^* Q^{-n} A \geqslant Q - \frac{1}{\gamma^n} \frac{\alpha^n (1-\alpha)}{v_Q} Q,$$

$$X_1 \geqslant Q(1-(1-\alpha)) = \alpha Q.$$

Thus $\alpha Q \leqslant X_1 \leqslant \beta Q$.
Assume $\alpha Q \leqslant X_s \leqslant \beta Q$. According to Theorem 2 we have

$$X_s^n \leqslant \beta^n v_Q Q^n,$$
$$X_{s+1} = Q - A^* X_s^{-n} A \leqslant Q - \frac{1}{\beta^n v_Q} A^* Q^{-n} A \leqslant \left(1 - \frac{1}{\beta^n v_Q} \beta^n v_Q (1-\beta)\right) Q = \beta Q.$$

Similarly

$$X_s^{-1} \leqslant \frac{1}{\alpha} Q^{-1},$$
$$X_s^{-n} \leqslant \frac{1}{\alpha^n} \left(\frac{M_{Q^{-1}}}{m_{Q^{-1}}}\right)^{n-1}, \; Q^{-n} = \frac{1}{\alpha^n} v_Q Q^{-n},$$
$$A^* X_s^{-n} A \leqslant \frac{v_Q}{\alpha^n} A^* Q^{-n} A,$$
$$X_{s+1} = Q - A^* X_s^{-n} A \geqslant \left(1 - \frac{v_Q}{\alpha^n} \frac{\alpha^n (1-\alpha)}{v_Q}\right) Q = \alpha Q.$$

Thus

$$\alpha Q \leqslant X_s \leqslant \beta Q, \quad s = 0, 1, 2, \ldots$$

We shall estimate difference between X_{s+p} and X_s:

$$\begin{aligned} X_{s+p} - X_s &= A^* X_{s+p-1}^{-n} (X_{s+p-1}^n - X_{s-1}^n) X_{s-1}^{-n} A \\ &= A^* \sum_{i=1}^{n} X_{s+p-1}^{-i} (X_{s+p-1} - X_{s-1}) X_{s-1}^{i-(n+1)} A \end{aligned}$$

for all integers $s, p = 1, 2, \ldots$. Since $\|X_j^{-k}\| \leqslant \frac{1}{\alpha^k}\|Q^{-1}\|^k$ for all $j, k = 1, \ldots, n$, we have

$$\begin{aligned}\|X_{s+p} - X_s\| &\leqslant \|A\|^2 \sum_{i=1}^{n} \|X_{s+p-1}^{-i}\| \|X_{s-1}^{i-(n+1)}\| \, \|X_{s+p-1} - X_{s-1}\| \\ &\leqslant \frac{n\|A\|^2\|Q^{-1}\|^{n+1}}{\alpha^{n+1}} \|X_{s+p-1} - X_{s-1}\| \leqslant \cdots \\ &\leqslant \left[\frac{n\|A\|^2\|Q^{-1}\|^{n+1}}{\alpha^{n+1}}\right]^s \|X_p - X_0\| = q^s \|X_p - X_0\|,\end{aligned}$$

where $q = \frac{n\|A\|^2\|Q^{-1}\|^{n+1}}{\alpha^{n+1}} < 1$. Thus

$$\|X_{s+p} - X_s\| \leqslant q^s \frac{1}{1-q} \|X_1 - X_0\|.$$

Hence the matrix sequence $\{X_s\}$ is a Cauchy sequence defined on the Banah space $[\alpha Q, \beta Q]$. This sequence is convergent and its limit $\hat{X} \in [\alpha Q, \beta Q] \subset ([n/(n+1)]Q, Q]$ is a positive definite solution of Eq. (1).

Assume that there are two solutions X' and X'' of Eq. (1) which belong on $([n/(n+1)]Q, Q]$. Then

$$\begin{aligned}\|X' - X''\| &\leqslant \|A\|^2 \sum_{i=1}^{n} \|(X')^{-i}\| \, \|(X'')^{i-(n+1)}\| \|X' - X''\| \\ &\leqslant q \, \|X' - X''\| < \|X' - X''\|.\end{aligned}$$

Hence $X' \equiv X''$, i.e. this solution $\hat{X}$ is unique on $([n/(n+1)]\, Q, Q]$. $\square$

Corollary 20. *If condition* (14) *holds and there exist reals* α *and* β, *for which* $n/(n+1) < \alpha \leqslant \beta \leqslant 1$ *and*

$$v_Q \beta^n (1-\beta) Q \leqslant A^* Q^{-n} A \leqslant \frac{1}{v_Q} \alpha^n (1-\alpha) Q,$$

then iterative procedure $X_{s+1} = Q - A^* X_s^{-n} A$ *with* $X_0 = \gamma Q\,(\alpha \leqslant \gamma \leqslant \beta)$ *converges to the special solution* X_1 *of* $X + A^* X^{-n} A = Q$.

Proof. First, if condition (14) holds, then for the factor q defined in Theorem 19 we have

$$q = \frac{n\|A\|^2\|Q^{-1}\|^{n+1}}{\alpha^{n+1}} < \frac{n}{\alpha^{n+1}} \frac{n^n}{(n+1)^{n+1}} < 1,$$

since $n/(n+1) < \alpha \leqslant 1$. Second, if condition (14) is true, then there exists a unique fixed point $\hat{Y}$ for $F(Y)$ such that $\hat{Y}^{-1}$ is a unique positive definite solution of (1) and $\hat{Y}$ has the corresponding property. If there are numbers α and β satisfying Theorem's 19 conditions, then there exists a unique positive definite solution $\tilde{X}$ of (1) and $[n/n+1]Q \leqslant \tilde{X}$. Thus $[n+1/n]Q^{-1} \geqslant \tilde{X}^{-1}$ and $\|\tilde{X}^{-1}\| \leqslant [n+1/n]\|Q^{-1}\|$. Hence, if (14) is true and conditions (i) and (i) of Theorem 19 are satisfied then $\tilde{X} \equiv \hat{Y}^{-1} \equiv X_1$. $\square$

3. Conclusion

We investigate the existence of the minimal solution X_S and the special solution X_1 of 1. We derive a new sufficient condition 4 for existence a minimal positive definite solution. If this condition is not satisfied, then iterative procedure 9 is considered. Under restrictions of Theorems 11 and 13 this method is convergent. We consider sufficient condition (14) for existence the special positive definite solution X_1 and iterative method (12) which converges to this solution under restrictions of Theorem 19.

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